

SUMS OF THE TRIPLE DIVISOR FUNCTION OVER VALUES OF A TERNARY QUADRATIC FORM

QINGFENG SUN AND DEYU ZHANG

ABSTRACT. Let $\tau_3(n)$ be the triple divisor function which is the number of solutions of the equation $d_1 d_2 d_3 = n$ in natural numbers. It is shown that

$$\sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau_3(n_1^2 + n_2^2 + n_3^2) = c_1 x^{\frac{3}{2}} (\log x)^2 + c_2 x^{\frac{3}{2}} \log x + c_3 x^{\frac{3}{2}} + O_\varepsilon(x^{\frac{11}{8} + \varepsilon})$$

for some constants c_1 , c_2 and c_3 .

CONTENTS

1. Introduction	1
2. Derivation of Theorem 1.1	4
3. Voronoi formula for the triple divisor function	7
4. Transformation of $\mathcal{S}(X)$	8
5. Contribution of \mathcal{B}_j , $1 \leq j \leq 4$	13
6. Contribution of \mathcal{B}_j , $5 \leq j \leq 13$	17
7. Computation of the main terms	19
8. Proof of Proposition 5.2	24
9. Estimation of the character sum $\mathcal{C}(b_1, b_2, b_3, n, m, v; q)$	26
References	29

1. INTRODUCTION

The divisor functions

$$\tau_k(n) = \sum_{\substack{d_1 \cdots d_k = n \\ d_1, \dots, d_k \in \mathbb{Z}^+}} 1,$$

are the basic arithmetic functions in number theory, with the generating Dirichlet series $\zeta^k(s)$ which are the simplest GL_k L -functions. While the Riemann zeta function $\zeta(s)$ has always been the most important and intensively studied L -function, the behavior of $\tau_k(n)$

Date: October 22, 2015.

Key words and phrases. Triple divisor function, ternary quadratic form, twisted character sum.

is far less than perfectly understood even for $k = 2$. For example, Hooley [6] proved that

$$\sum_{n \leq x} \tau(n^2 + a) = c_1 x \log x + c_2 x + O\left(x^{\frac{8}{9}} (\log x)^3\right) \quad (1.1)$$

for any fixed $a \in \mathbb{Z}$ such that $-a$ is not a perfect square, where c_1 and c_2 are constants depending only on a . Here as usual $\tau(n) := \tau_2(n)$. However, so far there are no asymptotic formulas for the sum $\sum_{n \leq x} \tau(f(n))$ for $f(x)$ of degree $\deg f \geq 3$. For the average behavior of the divisor functions over values of quadratic forms, Yu [15] proved that, as $x \rightarrow \infty$,

$$\sum_{1 \leq n_1, n_2 \leq \sqrt{x}} \tau(n_1^2 + n_2^2) = c_3 x \log x + c_4 x + O_\varepsilon\left(x^{\frac{3}{4} + \varepsilon}\right), \quad (1.2)$$

where c_3 and c_4 are constants. Calderón and de Velasco [1] studied the average behavior of $\tau(n)$ over values of ternary quadratic form and established the asymptotic formula

$$\sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau(n_1^2 + n_2^2 + n_3^2) = \frac{4\zeta(3)}{5\zeta(5)} x^{\frac{3}{2}} \log x + O(x^{\frac{3}{2}}). \quad (1.3)$$

Recently, Guo and Zhai improved (1.3) by showing that

$$\sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau(n_1^2 + n_2^2 + n_3^2) = \frac{4\zeta(3)}{5\zeta(5)} x^{\frac{3}{2}} \log x + c_5 x^{\frac{3}{2}} + O_\varepsilon(x^{\frac{4}{3}}), \quad (1.4)$$

where c_5 is a constant. The error term in (1.4) was further improved by Zhao [16] to $O(x \log x)$. Nothing of type (1.1), (1.2) or (1.3) is known for $\tau_k(n)$ with $k \geq 3$ and in fact the situation becomes even more difficult for $k \geq 3$ if one considers the sum

$$\sum_{n \leq x} a_n \tau_k(n)$$

for various sparse arithmetic sequences a_n . There are few results in this direction. For $\tau_3(n)$, Friedlander and Iwaniec [3] showed that, for $x \geq 3$,

$$\sum_{\substack{n_1^2 + n_2^2 \leq x \\ (n_1, n_2) = 1}} \tau_3(n_1^2 + n_2^2) = cx^{\frac{2}{3}} (\log x)^2 + O\left(x^{\frac{2}{3}} (\log x)^{\frac{7}{4}} (\log \log x)^{\frac{1}{2}}\right),$$

where c is a constant.

In this paper, we want to prove an asymptotic formula of type (1.4) for $\tau_3(n)$. Our main result is the following theorem.

Theorem 1.1. *For any $x \geq x_0$ (x_0 is a large constant) and any $\varepsilon > 0$, we have*

$$\begin{aligned} \sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau_3(n_1^2 + n_2^2 + n_3^2) &= \frac{\mathcal{C}_0 \mathcal{J}_0}{4} x^{\frac{3}{2}} (\log x)^2 + \frac{1}{2} (\mathcal{C}_1 \mathcal{J}_0 + \mathcal{C}_0 \mathcal{J}_1) x^{\frac{3}{2}} \log x \\ &\quad + \frac{1}{2} \left(\mathcal{C}_2 \mathcal{J}_0 + \mathcal{C}_1 \mathcal{J}_1 + \frac{1}{2} \mathcal{C}_0 \mathcal{J}_2 \right) x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{11}{8} + \varepsilon}\right), \end{aligned}$$

where for $\ell = 0, 1, 2$,

$$\mathcal{J}_\ell = \int_{-\infty}^{\infty} \left(\int_0^3 (\log u)^\ell e(-\beta u) du \right) \left(\int_0^1 e(\beta v^2) dv \right)^3 d\beta$$

and

$$\mathcal{C}_\ell = \sum_{q=1}^{\infty} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_\ell(n, q) \sum_{\substack{a=1 \\ (a, q)=1}}^q G(a, 0; q)^3 S(-\bar{a}, 0; q).$$

Here \bar{a} denotes the multiplicative inverse of $a \bmod q$, $S(a, b; c)$ is the classical Kloosterman sum, $G(a, b; q)$ is the Gauss sum

$$G(a, b; q) = \sum_{d \bmod q} e \left(\frac{ad^2 + bd}{q} \right),$$

and $P_j(n, q)$ ($j = 1, 2$) are given by

$$\begin{aligned} P_1(n, q) &= \frac{5}{3} \log n - 3 \log q + 3\gamma - \frac{1}{3\tau(n)} \sum_{d|n} \log d, \\ P_2(n, q) &= (\log n)^2 - 5 \log q \log n + \frac{9}{2} (\log q)^2 + 3\gamma^2 - 3\gamma_1 + 7\gamma \log n - 9\gamma \log q \\ &\quad + \frac{1}{\tau(n)} \left((\log n + \log q - 5\gamma) \sum_{d|n} \log d - \frac{3}{2} \sum_{d|n} (\log d)^2 \right) \end{aligned}$$

with $\gamma := \lim_{s \rightarrow 1} (\zeta(s) - \frac{1}{s-1})$ being the Euler constant and $\gamma_1 := -\frac{d}{ds} (\zeta(s) - \frac{1}{s-1}) \big|_{s=1}$ being the Stieltjes constant.

A similar asymptotic formula can be derived for the slightly modified sum

$$\sum_{\substack{1 \leq n_1^2 + n_2^2 + n_3^2 \leq x \\ (n_1, n_2, n_3) \in \mathbb{Z}^3}} \tau_3(n_1^2 + n_2^2 + n_3^2)$$

which is in some sense simpler than the sum in Theorem 1.1, since obviously we have

$$\sum_{1 \leq n_1^2 + n_2^2 + n_3^2 \leq x} \tau_3(n_1^2 + n_2^2 + n_3^2) = \sum_{1 \leq n \leq x} \tau_3(n) r_3(n), \quad (1.5)$$

where

$$r_3(n) = \# \{ (n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1^2 + n_2^2 + n_3^2 = n \}.$$

Theorem 1.2. For any $x \geq x_0$ (x_0 is the same as that in Theorem 1.1) and any $\varepsilon > 0$, we have

$$\sum_{\substack{1 \leq n_1^2 + n_2^2 + n_3^2 \leq x \\ (n_1, n_2, n_3) \in \mathbb{Z}^3}} \tau_3(n_1^2 + n_2^2 + n_3^2) = 2\mathcal{C}_0\mathcal{K}_0 x^{\frac{3}{2}} (\log x)^2 + 4(\mathcal{C}_1\mathcal{K}_0 + \mathcal{C}_0\mathcal{K}_1) x^{\frac{3}{2}} \log x \\ + 4 \left(\mathcal{C}_2\mathcal{K}_0 + \mathcal{C}_1\mathcal{K}_1 + \frac{1}{2}\mathcal{C}_0\mathcal{K}_2 \right) x^{\frac{3}{2}} + O_\varepsilon \left(x^{\frac{11}{8} + \varepsilon} \right),$$

where \mathcal{C}_ℓ 's are as in Theorem 1.1 and

$$\mathcal{K}_\ell = \int_{-\infty}^{\infty} \left(\int_0^1 (\log u)^\ell e(-\beta u) du \right) \left(\int_0^1 e(\beta v^2) dv \right)^3 d\beta.$$

Remark 1. In Theorems 1.1 and 1.2, x_0 is an absolute constant which can be explicitly computed. We can take $x_0 = 4^8$.

Remark 2. The proofs of Theorems 1.1 and 1.2 are partly similar as that in Sun [12] and Zhao [16]. The main saving comes from square-root cancelation of a two dimensional twisted character sum (see Section 9) which benefits from the beautiful theorem of Fu [2].

Remark 3. The proof of Theorem 1.2 is similar as that of Theorem 1.1 and we shall omit it for simplicity. In view of (1.5), the error term in Theorem 1.2 may be further improved by appealing to the analytic properties of the L -function

$$\sum_{n \geq 1} \frac{\tau_3(n) r_3(n)}{n^s}.$$

However, we will not take up this issue in this paper.

Notation. Throughout the paper, the letters q , m and n , with or without subscript, denote integers. The letter ε is an arbitrarily small positive constant, not necessarily the same at different occurrences. The symbol $\ll_{a,b,c}$ denotes that the implied constant depends at most on a , b and c .

2. DERIVATION OF THEOREM 1.1

Let \mathcal{V} denote the set $[1, \sqrt{x}] \cap \mathbb{Z}$ and $r_3^*(n) = \sum_{\substack{n_1^2 + n_2^2 + n_3^2 = n \\ (n_1, n_2, n_3) \in \mathcal{V}^3}} 1$. By dyadic subdivision, we

decompose the aimed sum into partial sums

$$\sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau_3(n_1^2 + n_2^2 + n_3^2) = \sum_{j \geq 1} \sum_{3x/2^j < n \leq 3x/2^{j-1}} \tau_3(n) r_3^*(n). \quad (2.1)$$

Notice that the inner sum in (2.1) vanishes for $j > \log 6x / \log 2$. So we are treating $O(\log x)$ sums of the form

$$\sum_{X_j/2 < n \leq X_j} \tau_3(n) r_3^*(n),$$

where $X_j = 3x/2^{j-1}$. Let $\phi(y)$ be a smooth function supported on $[1/2, 1]$, identically equal 1 on $[1/2 + M^{-1}, 1 - M^{-1}]$ with $M > 4$, and satisfy $\phi^{(j)}(y) \ll_j M^j$ for any integer $j \geq 0$. Then we have

$$\sum_{X_j/2 < n \leq X_j} \tau_3(n) r_3^*(n) = \sum_{n \geq 1} \tau_3(n) r_3^*(n) \phi\left(\frac{n}{X_j}\right) + O_\varepsilon(X_j^{\frac{3}{2}+\varepsilon} M^{-1}). \quad (2.2)$$

Here the O -term comes from the bounds $\tau_3(n) \ll_\varepsilon n^\varepsilon$ and

$$r_3^*(n) \leq r_3(n) = \sum_{\substack{n_1^2 + n_2^2 + n_3^2 = n \\ (n_1, n_2, n_3) \in \mathbb{Z}^3}} 1 \ll_\varepsilon n^{\frac{1}{2}+\varepsilon}.$$

Thus it remains to study the smoothed sum

$$\mathcal{S}(X) = \sum_{n \geq 1} \tau_3(n) r_3^*(n) \phi\left(\frac{n}{X}\right).$$

We are going to prove the following asymptotic formula for $\mathcal{S}(X)$.

Theorem 2.1. *For any $1 < X \leq 3x$ and any $\varepsilon > 0$, we have*

$$\mathcal{S}(X) = \frac{1}{2} \mathcal{I}_0(X) \mathcal{C}_2 x^{\frac{3}{2}} + \frac{1}{2} \mathcal{I}_1(X) \mathcal{C}_1 x^{\frac{3}{2}} + \frac{1}{4} \mathcal{I}_2(X) \mathcal{C}_0 x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon} M + x^{\frac{3}{2}+\varepsilon} M^{-1}\right),$$

where

$$\mathcal{C}_\ell = \sum_{q=1}^{\infty} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_\ell(n, q) \sum_{a=1}^q{}^* G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right), \quad (2.3)$$

and

$$\mathcal{I}_\ell(X) = \int_{-\infty}^{\infty} \left(\int_{X/2}^X e(-\beta u) (\log u)^\ell du \right) \left(\int_0^1 e(\beta x v^2) dv \right)^3 d\beta.$$

We set the proof of Theorem 2.1 aside and continue the derivation of Theorem 1.1. Applying Theorem 2.1 to the sum on the right side of (2.2) and taking $M = x^{\frac{1}{8}}$ we get

$$\sum_{X_j/2 < n \leq X_j} \tau_3(n) r_3^*(n) = \frac{1}{2} \mathcal{I}_0(X_j) \mathcal{C}_2 x^{\frac{3}{2}} + \frac{1}{2} \mathcal{I}_1(X_j) \mathcal{C}_1 x^{\frac{3}{2}} + \frac{1}{4} \mathcal{I}_2(X_j) \mathcal{C}_0 x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{11}{8}+\varepsilon}\right). \quad (2.4)$$

Set $j_0 := j_0(x) = \lceil \log x / \log 2 \rceil$. Plugging (2.4) into (2.1) we obtain

$$\begin{aligned}
& \sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau_3(n_1^2 + n_2^2 + n_3^2) \\
&= \sum_{1 \leq j \leq j_0} \sum_{3x/2^j < n \leq 3x/2^{j-1}} \tau_3(n) r_3^*(n) + O(1) \\
&= \frac{1}{2} \mathcal{C}_2 x^{\frac{3}{2}} \sum_{1 \leq j \leq j_0} \mathcal{I}_0 \left(\frac{3x}{2^{j-1}} \right) + \frac{1}{2} \mathcal{C}_1 x^{\frac{3}{2}} \sum_{1 \leq j \leq j_0} \mathcal{I}_1 \left(\frac{3x}{2^{j-1}} \right) + \frac{1}{4} \mathcal{C}_0 x^{\frac{3}{2}} \sum_{1 \leq j \leq j_0} \mathcal{I}_2 \left(\frac{3x}{2^{j-1}} \right) \\
&\quad + O_\varepsilon \left(x^{\frac{11}{8} + \varepsilon} \right),
\end{aligned} \tag{2.5}$$

where for $\ell = 0, 1, 2$,

$$\begin{aligned}
\sum_{1 \leq j \leq j_0} \mathcal{I}_\ell \left(\frac{3x}{2^{j-1}} \right) &= \int_{-\infty}^{\infty} \left(\sum_{1 \leq j \leq j_0} \int_{\frac{3x}{2^j}}^{\frac{3x}{2^{j-1}}} (\log u)^\ell e(-\beta u) du \right) \left(\int_0^1 e(\beta x v^2) dv \right)^3 d\beta \\
&= x \int_{-\infty}^{\infty} \left(\int_{\frac{3}{2^{j_0}}}^3 (\log ux)^\ell e(-\beta xu) du \right) \left(\int_0^1 e(\beta x v^2) dv \right)^3 d\beta \\
&= \sum_{i=0}^{\ell} C_\ell^i (\log x)^{\ell-i} \int_{-\infty}^{\infty} \left(\int_0^3 (\log u)^i e(-\beta u) du \right) \left(\int_0^1 e(\beta v^2) dv \right)^3 d\beta \\
&\quad + \mathcal{R}_\ell(x),
\end{aligned} \tag{2.6}$$

and for $x \geq 6$,

$$\begin{aligned}
\mathcal{R}_\ell(x) &= \sum_{i=0}^{\ell} C_\ell^i (\log x)^{\ell-i} \int_{-\infty}^{\infty} \left(\int_0^{\frac{3}{2^{j_0}}} (\log u)^i e(-\beta u) du \right) \left(\int_0^1 e(\beta v^2) dv \right)^3 d\beta \\
&\ll_\ell (\log x)^\ell \int_0^{\frac{3}{2^{j_0}}} (-\log u)^\ell du \\
&\ll_\ell x^{-1} (\log x)^4.
\end{aligned} \tag{2.7}$$

By (2.5)-(2.7) we obtain

$$\begin{aligned}
\sum_{1 \leq n_1, n_2, n_3 \leq \sqrt{x}} \tau_3(n_1^2 + n_2^2 + n_3^2) &= \frac{\mathcal{C}_0 \mathcal{J}_0}{4} x^{\frac{3}{2}} (\log x)^2 + \frac{1}{2} (\mathcal{C}_1 \mathcal{J}_0 + \mathcal{C}_0 \mathcal{J}_1) x^{\frac{3}{2}} \log x \\
&\quad + \frac{1}{2} \left(\mathcal{C}_2 \mathcal{J}_0 + \mathcal{C}_1 \mathcal{J}_1 + \frac{1}{2} \mathcal{C}_0 \mathcal{J}_2 \right) x^{\frac{3}{2}} + O_\varepsilon \left(x^{\frac{11}{8} + \varepsilon} \right),
\end{aligned}$$

where for \mathcal{C}_ℓ is defined in (2.3) and

$$\mathcal{J}_\ell = \int_{-\infty}^{\infty} \left(\int_0^3 (\log u)^\ell e(-\beta u) du \right) \left(\int_0^1 e(\beta v^2) dv \right)^3 d\beta.$$

This finishes the proof of Theorem 1.1. The following sections are devoted to the proof of Theorem 2.1.

3. VORONOI FORMULA FOR THE TRIPLE DIVISOR FUNCTION

The Voronoi formula for $\tau_3(n)$ was first proved by Ivić [8] and later in [11], Li derived a more explicit formula. To adopt Li's result, set

$$\sigma_{0,0}(k, l) = \sum_{\substack{d_1 | l \\ d_1 > 0}} \sum_{\substack{d_2 | \frac{l}{d_1} \\ d_2 > 0 \\ (d_2, k) = 1}} 1. \quad (3.1)$$

Let $\zeta(s)$ be the Riemann zeta function, $\gamma := \lim_{s \rightarrow 1} (\zeta(s) - \frac{1}{s-1})$ be the Euler constant and $\gamma_1 := -\frac{d}{ds} (\zeta(s) - \frac{1}{s-1}) \Big|_{s=1}$ be the Stieltjes constant. For $\phi(y) \in C_c(0, \infty)$, $k = 0, 1$ and $\sigma > -1 - 2k$, set

$$\Phi_k(y) = \frac{1}{2\pi i} \int_{\text{Re}(s)=\sigma} (\pi^3 y)^{-s} \frac{\Gamma\left(\frac{1+s+2k}{2}\right)^3}{\Gamma\left(\frac{-s}{2}\right)^3} \tilde{\phi}(-s-k) ds$$

with $\tilde{\phi}(s) = \int_0^\infty \phi(u) u^{s-1} du$ the Mellin transform of ϕ , and

$$\Phi^\pm(y) = \Phi_0(y) \pm \frac{1}{i\pi^3 y} \Phi_1(y).$$

Lemma 3.1. *For $\phi(y) \in C_c^\infty(0, \infty)$, $a, \bar{a}, q \in \mathbb{Z}^+$ with $a\bar{a} \equiv 1 \pmod{q}$, we have*

$$\begin{aligned} & \sum_{n \geq 1} \tau_3(n) e\left(\frac{an}{q}\right) \phi(n) \\ &= \frac{q}{2\pi^{\frac{3}{2}}} \sum_{\pm} \sum_{n|q} \sum_{m \geq 1} \frac{1}{nm} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) S\left(\pm m, \bar{a}; \frac{q}{n}\right) \Phi^\pm\left(\frac{mn^2}{q^3}\right) \\ &+ \frac{1}{2q^2} \tilde{\phi}(1) \sum_{n|q} n\tau(n) P_2(n, q) S\left(0, \bar{a}; \frac{q}{n}\right) \\ &+ \frac{1}{2q^2} \tilde{\phi}'(1) \sum_{n|q} n\tau(n) P_1(n, q) S\left(0, \bar{a}; \frac{q}{n}\right) \\ &+ \frac{1}{4q^2} \tilde{\phi}''(1) \sum_{n|q} n\tau(n) S\left(0, \bar{a}; \frac{q}{n}\right), \end{aligned}$$

where $S(a, b; c)$ is the classical Kloosterman sum,

$$P_1(n, q) = \frac{5}{3} \log n - 3 \log q + 3\gamma - \frac{1}{3\tau(n)} \sum_{d|n} \log d, \quad (3.2)$$

and

$$\begin{aligned}
P_2(n, q) &= (\log n)^2 - 5 \log q \log n + \frac{9}{2} (\log q)^2 + 3\gamma^2 - 3\gamma_1 + 7\gamma \log n - 9\gamma \log q \\
&\quad + \frac{1}{\tau(n)} \left((\log n + \log q - 5\gamma) \sum_{d|n} \log d - \frac{3}{2} \sum_{d|n} (\log d)^2 \right). \tag{3.3}
\end{aligned}$$

The functions $\Phi^\pm(y)$ have the following properties (see Sun [12]).

Lemma 3.2. *Suppose that $\phi(y)$ is a smooth function of compact support in $[AX, BX]$, where $X > 0$ and $B > A > 0$, satisfying $\phi^{(j)}(y) \ll_{A,B,j} P^j$ for any integer $j \geq 0$. Then for $y > 0$ and any integer $\ell \geq 0$, we have*

$$\Phi^\pm(y) \ll_{A,B,\ell,\varepsilon} (yX)^{-\varepsilon} (PX)^3 \left(\frac{y}{P^3 X^2} \right)^{-\ell}.$$

By Lemma 3.2, for any fixed $\varepsilon > 0$ and $yX \geq X^\varepsilon (PX)^3$, $\Phi^\pm(y)$ are negligibly small. Moreover, for $yX \gg X^\varepsilon$, we have an asymptotic formula for $\Phi_k(y)$ (see [8], [10], [14]).

Lemma 3.3. *Suppose that $\phi(y)$ is a smooth function of compact support on $[AX, BX]$, where $X > 0$ and $B > A > 0$. Then for $y > 0$, $yX \gg 1$, $\ell \geq 2$ and $k = 0, 1$, we have*

$$\begin{aligned}
\Phi_k(y) &= (\pi^3 y)^{k+1} \sum_{j=1}^{\ell} \int_0^\infty \phi(u) \left(a_k(j) e \left(3(yu)^{\frac{1}{3}} \right) + b_k(j) e \left(-3(yu)^{\frac{1}{3}} \right) \right) \frac{du}{(\pi^3 y u)^{\frac{2}{3}}} \\
&\quad + O_{A,B,\varepsilon,\ell} \left((\pi^3 y)^k (\pi^3 y X)^{-\frac{\ell}{3} + \frac{1}{2} + \varepsilon} \right),
\end{aligned}$$

where $a_k(j)$, $b_k(j)$ are constants with

$$a_0(1) = -\frac{2\sqrt{3\pi}}{6\pi i}, \quad b_0(1) = \frac{2\sqrt{3\pi}}{6\pi i}, \quad a_1(1) = b_1(1) = -\frac{2\sqrt{3\pi}}{6\pi}.$$

4. TRANSFORMATION OF $\mathcal{S}(X)$

Applying the circle method, we have

$$\mathcal{S}(X) = \int_0^1 \mathcal{F}^3(\alpha) \mathcal{G}(\alpha) d\alpha,$$

where

$$\mathcal{F}(\alpha) = \sum_{n \in \mathcal{V}} e(\alpha n^2),$$

and

$$\mathcal{G}(\alpha) = \sum_{n \geq 1} \tau_3(n) e(-\alpha n) \phi \left(\frac{n}{X} \right). \tag{4.1}$$

Note that $\mathcal{F}^3(\alpha)\mathcal{G}(\alpha)$ is a periodic function of period 1. We have

$$\mathcal{S}(X) = \int_{-1/(Q+1)}^{Q/(Q+1)} \mathcal{F}^3(\alpha)\mathcal{G}(\alpha)d\alpha,$$

where Q is a large integer to be chosen later. Then we can evaluate $\mathcal{S}(X)$ by dissecting the interval $(-1/(Q+1), Q/(Q+1)]$ with Farey's points of order Q (see for example Iwaniec [9]). Let $\frac{a'}{q'} < \frac{a}{q} < \frac{a''}{q''}$ be adjacent points, which are determined by the conditions

$$Q < q + q', q + q'' \leq q + Q, \quad aq' \equiv 1(\text{mod } q), \quad aq'' \equiv -1(\text{mod } q).$$

Then

$$\left(\frac{-1}{Q+1}, \frac{Q}{Q+1} \right] = \bigcup_{\substack{0 \leq a < q \leq Q \\ (a, q) = 1}} \left(\frac{a}{q} - \frac{1}{q(q+q')}, \frac{a}{q} + \frac{1}{q(q+q'')} \right].$$

It follows that

$$\mathcal{S}(X) = \sum_{q \leq Q} \sum_{a=1}^q \int_{\mathcal{M}(a, q)} \mathcal{F}^3\left(\frac{a}{q} + \beta\right) \mathcal{G}\left(\frac{a}{q} + \beta\right) d\beta,$$

where $*$ denotes the condition $(a, q) = 1$ and

$$\mathcal{M}(a, q) = \left(-\frac{1}{q(q+q')}, \frac{1}{q(q+q'')} \right].$$

Exchanging the order of the summation over a and the integration over β as in Heath-Brown [5], we have

$$\mathcal{S}(X) = \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \text{ mod } q} \varrho(v, q, \beta) \sum_{a=1}^q e\left(-\frac{\bar{a}v}{q}\right) \mathcal{F}^3\left(\frac{a}{q} + \beta\right) \mathcal{G}\left(\frac{a}{q} + \beta\right) d\beta \quad (4.2)$$

where $\varrho(v, q, \beta)$ satisfies

$$\varrho(v, q, \beta) \ll \frac{1}{1 + |v|}. \quad (4.3)$$

For an asymptotic formula of $\mathcal{F}\left(\frac{a}{q} + \beta\right)$, we quote the following result (see Theorem 4.1 in [13] or Lemma 4.1 in [16]).

Lemma 4.1. *Let $Q = [5\sqrt{x}]$. Suppose that $(a, q) = 1$, $q \leq Q$ and $|\beta| \leq 1/(qQ)$. We have*

$$\mathcal{F}\left(\frac{a}{q} + \beta\right) = \frac{G(a, 0; q)}{q} \Psi_0(\beta) + \sum_{-\frac{3q}{2} < b \leq \frac{3q}{2}} G(a, b; q) \Psi(b, q, \beta), \quad (4.4)$$

where $G(a, b; q)$ is the Gauss sum

$$G(a, b; q) = \sum_{d \text{ mod } q} e\left(\frac{ad^2 + bd}{q}\right), \quad (4.5)$$

$\Psi_0(\beta)$ is the integral

$$\Psi_0(\beta) = \int_0^{\sqrt{x}} e(\beta u^2) du, \quad (4.6)$$

and $\Psi(b, q, \beta)$ satisfies

$$\sum_{-\frac{3q}{2} < b \leq \frac{3q}{2}} |\Psi(b, q, \beta)| \ll \log(q+2). \quad (4.7)$$

For $\mathcal{G}(\alpha)$ in (4.1), we apply Lemma 3.1 with $\phi_\beta(y) = \phi\left(\frac{y}{X}\right) e(-\beta y)$ getting

$$\begin{aligned} \mathcal{G}\left(\frac{a}{q} + \beta\right) &= \sum_{n \geq 1} \tau_3(n) e\left(-\frac{an}{q}\right) \phi_\beta(n) \\ &= \frac{q}{2\pi^{\frac{3}{2}}} \sum_{\pm} \sum_{n|q} \sum_{m \geq 1} \frac{1}{nm} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0}\left(\frac{n}{n_1 n_2}, m\right) S\left(\pm m, -\bar{a}; \frac{q}{n}\right) \Phi_\beta^\pm\left(\frac{mn^2}{q^3}\right) \\ &\quad + \frac{1}{2q^2} \tilde{\phi}_\beta(1) \sum_{n|q} n \tau(n) P_2(n, q) S\left(0, -\bar{a}; \frac{q}{n}\right) \\ &\quad + \frac{1}{2q^2} \tilde{\phi}'_\beta(1) \sum_{n|q} n \tau(n) P_1(n, q) S\left(0, -\bar{a}; \frac{q}{n}\right) \\ &\quad + \frac{1}{4q^2} \tilde{\phi}''_\beta(1) \sum_{n|q} n \tau(n) S\left(0, -\bar{a}; \frac{q}{n}\right), \end{aligned} \quad (4.8)$$

where

$$\Phi_\beta^\pm(y) = \Phi_0(y, \beta) \pm \frac{1}{i\pi^3 y} \Phi_1(y, \beta) \quad (4.9)$$

with

$$\Phi_k(y, \beta) = \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=\sigma} (\pi^3 y)^{-s} \frac{\Gamma\left(\frac{1+s+2k}{2}\right)^3}{\Gamma\left(\frac{-s}{2}\right)^3} \tilde{\phi}_\beta(-s-k) ds, \quad (4.10)$$

and $P_j(n, q)$ ($j = 1, 2$) defined in (3.2) and (3.3).

By (4.4) and (4.8), we have

$$\sum_{a=1}^q e\left(-\frac{\bar{a}v}{q}\right) \mathcal{F}^3\left(\frac{a}{q} + \beta\right) \mathcal{G}\left(\frac{a}{q} + \beta\right) = \sum_{j=1}^{16} \mathcal{B}_j(v, q, \beta), \quad (4.11)$$

where

$$\begin{aligned}\mathcal{B}_1(v, q, \beta) &= \frac{q}{2\pi^{\frac{3}{2}}} \sum_{\pm} \sum_{n|q} \sum_{m \geq 1} \frac{1}{nm} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \Phi_{\beta}^{\pm} \left(\frac{mn^2}{q^3} \right) \\ &\times \sum_{\substack{-\frac{3q}{2} < b_j \leq \frac{3q}{2} \\ 1 \leq j \leq 3}} \Psi(b_1, q, \beta) \Psi(b_2, q, \beta) \Psi(b_3, q, \beta) \mathcal{C}(b_1, b_2, b_3, n, m, v; q) \quad (4.12)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_2(v, q, \beta) &= \frac{3}{2\pi^{\frac{3}{2}}} \Psi_0(\beta) \sum_{\pm} \sum_{n|q} \sum_{m \geq 1} \frac{1}{nm} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \Phi_{\beta}^{\pm} \left(\frac{mn^2}{q^3} \right) \\ &\times \sum_{\substack{-\frac{3q}{2} < b_j \leq \frac{3q}{2} \\ j=1,2}} \Psi(b_1, q, \beta) \Psi(b_2, q, \beta) \mathcal{C}(0, b_1, b_2, n, m, v; q), \quad (4.13)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_3(v, q, \beta) &= \frac{3}{2\pi^{\frac{3}{2}}} \frac{\Psi_0(\beta)^2}{q} \sum_{\pm} \sum_{n|q} \sum_{m \geq 1} \frac{1}{nm} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \Phi_{\beta}^{\pm} \left(\frac{mn^2}{q^3} \right) \\ &\times \sum_{-\frac{3q}{2} < b \leq \frac{3q}{2}} \Psi(b, q, \beta) \mathcal{C}(0, 0, b, n, m, v; q), \quad (4.14)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_4(v, q, \beta) &= \frac{1}{2\pi^{\frac{3}{2}}} \frac{\Psi_0(\beta)^3}{q^2} \sum_{\pm} \sum_{n|q} \sum_{m \geq 1} \frac{1}{nm} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \Phi_{\beta}^{\pm} \left(\frac{mn^2}{q^3} \right) \\ &\times \mathcal{C}(0, 0, 0, n, m, v; q), \quad (4.15)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_5(v, q, \beta) &= \frac{1}{2} \frac{\tilde{\phi}_{\beta}(1)}{q^2} \sum_{n|q} n \tau(n) P_2(n, q) \sum_{\substack{-\frac{3q}{2} < b_j \leq \frac{3q}{2} \\ 1 \leq j \leq 3}} \Psi(b_1, q, \beta) \Psi(b_2, q, \beta) \Psi(b_3, q, \beta) \\ &\times \mathcal{C}(b_1, b_2, b_3, n, 0, v; q), \quad (4.16)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_6(v, q, \beta) &= \frac{1}{2} \frac{\tilde{\phi}'_{\beta}(1)}{q^2} \sum_{n|q} n \tau(n) P_1(n, q) \sum_{\substack{-\frac{3q}{2} < b_j \leq \frac{3q}{2} \\ 1 \leq j \leq 3}} \Psi(b_1, q, \beta) \Psi(b_2, q, \beta) \Psi(b_3, q, \beta) \\ &\times \mathcal{C}(b_1, b_2, b_3, n, 0, v; q), \quad (4.17)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_7(v, q, \beta) &= \frac{1}{4} \frac{\tilde{\phi}''_{\beta}(1)}{q^2} \sum_{n|q} n \tau(n) \sum_{\substack{-\frac{3q}{2} < b_j \leq \frac{3q}{2} \\ 1 \leq j \leq 3}} \Psi(b_1, q, \beta) \Psi(b_2, q, \beta) \Psi(b_3, q, \beta) \\ &\times \mathcal{C}(b_1, b_2, b_3, n, 0, v; q), \quad (4.18)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_8(v, q, \beta) &= \frac{3}{2} \frac{\tilde{\phi}_{\beta}(1) \Psi_0(\beta)}{q^3} \sum_{n|q} n \tau(n) P_2(n, q) \sum_{\substack{-\frac{3q}{2} < b_j \leq \frac{3q}{2} \\ j=1,2}} \Psi(b_1, q, \beta) \Psi(b_2, q, \beta) \\ &\times \mathcal{C}(0, b_1, b_2, n, 0, v; q), \quad (4.19)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_9(v, q, \beta) &= \frac{3}{2} \frac{\tilde{\phi}'_\beta(1) \Psi_0(\beta)}{q^3} \sum_{n|q} n \tau(n) P_1(n, q) \sum_{\substack{-\frac{3q}{2} < b_j \leq \frac{3q}{2} \\ j=1,2}} \Psi(b_1, q, \beta) \Psi(b_2, q, \beta) \\ &\quad \times \mathcal{C}(0, b_1, b_2, n, 0, v; q),\end{aligned}\tag{4.20}$$

$$\begin{aligned}\mathcal{B}_{10}(v, q, \beta) &= \frac{3}{4} \frac{\tilde{\phi}''_\beta(1) \Psi_0(\beta)}{q^3} \sum_{n|q} n \tau(n) \sum_{\substack{-\frac{3q}{2} < b_j \leq \frac{3q}{2} \\ j=1,2}} \Psi(b_1, q, \beta) \Psi(b_2, q, \beta) \\ &\quad \times \mathcal{C}(0, b_1, b_2, n, 0, v; q),\end{aligned}\tag{4.21}$$

$$\begin{aligned}\mathcal{B}_{11}(v, q, \beta) &= \frac{3}{2} \frac{\tilde{\phi}_\beta(1) \Psi_0(\beta)^2}{q^4} \sum_{n|q} n \tau(n) P_2(n, q) \sum_{-\frac{3q}{2} < b \leq \frac{3q}{2}} \Psi(b, q, \beta) \\ &\quad \times \mathcal{C}(0, 0, b, n, 0, v; q),\end{aligned}\tag{4.22}$$

$$\begin{aligned}\mathcal{B}_{12}(v, q, \beta) &= \frac{3}{2} \frac{\tilde{\phi}'_\beta(1) \Psi_0(\beta)^2}{q^4} \sum_{n|q} n \tau(n) P_1(n, q) \sum_{-\frac{3q}{2} < b \leq \frac{3q}{2}} \Psi(b, q, \beta) \\ &\quad \times \mathcal{C}(0, 0, b, n, 0, v; q),\end{aligned}\tag{4.23}$$

$$\begin{aligned}\mathcal{B}_{13}(v, q, \beta) &= \frac{3}{4} \frac{\tilde{\phi}''_\beta(1) \Psi_0(\beta)^2}{q^4} \sum_{n|q} n \tau(n) \sum_{-\frac{3q}{2} < b \leq \frac{3q}{2}} \Psi(b, q, \beta) \\ &\quad \times \mathcal{C}(0, 0, b, n, 0, v; q),\end{aligned}\tag{4.24}$$

$$\mathcal{B}_{14}(v, q, \beta) = \frac{1}{2} \frac{\tilde{\phi}_\beta(1) \Psi_0(\beta)^3}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \mathcal{C}(0, 0, 0, n, 0, v; q),\tag{4.25}$$

$$\mathcal{B}_{15}(v, q, \beta) = \frac{1}{2} \frac{\tilde{\phi}'_\beta(1) \Psi_0(\beta)^3}{q^5} \sum_{n|q} n \tau(n) P_1(n, q) \mathcal{C}(0, 0, 0, n, 0, v; q),\tag{4.26}$$

$$\mathcal{B}_{16}(v, q, \beta) = \frac{1}{4} \frac{\tilde{\phi}''_\beta(1) \Psi_0(\beta)^3}{q^5} \sum_{n|q} n \tau(n) \mathcal{C}(0, 0, 0, n, 0, v; q)\tag{4.27}$$

with

$$\mathcal{C}(b_1, b_2, b_3, n, m, v; q) = \sum_{a=1}^q e\left(\frac{-\bar{a}v}{q}\right) G(a, b_1; q) G(a, b_2; q) G(a, b_3; q) S\left(-\bar{a}, \pm m; \frac{q}{n}\right).\tag{4.28}$$

We will show that \mathcal{B}_j , $1 \leq j \leq 13$, contribute the remainder terms, and \mathcal{B}_j , $14 \leq j \leq 16$, contribute the main terms.

5. CONTRIBUTION OF \mathcal{B}_j , $1 \leq j \leq 4$

The estimation of \mathcal{B}_j , $1 \leq j \leq 4$, are similar as the arguments in [12]. Since the cancelation from the character sums $\mathcal{C}(b_1, b_2, b_3, n, m, v; q)$ is the main saving for our final result, we first have the following proposition.

Proposition 5.1. *Let $q = q_1 q_0 q'_3$, $q_1 | n$, $4q_0$ square-full and q'_3 square-free. For any $\varepsilon > 0$, we have*

$$\mathcal{C}(b_1, b_2, b_3, n, m, v; q) \ll_{\varepsilon} \frac{(q_1 q_0)^{3+\varepsilon} q_3'^{\frac{5}{2}+\varepsilon}}{\sqrt{n}}. \quad (5.1)$$

Next, we need the following result which will be proved in Section 8.

Proposition 5.2. *For any $\varepsilon > 0$, we have*

$$\sum_{\pm} \sum_{m \geq 1} \frac{1}{m} \sum_{n_1 | n} \sum_{n_2 | \frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \left| \Phi_{\beta}^{\pm} \left(\frac{mn^2}{q^3} \right) \right| \ll_{\varepsilon} X^{\varepsilon} n^{\varepsilon} (M + |\beta|^2 X^2).$$

By the second derivative test and the trivial estimation, $\Psi_0(\beta)$ in (4.6) is bounded by

$$\Psi_0(\beta) \ll \left(\frac{x}{1 + |\beta|x} \right)^{\frac{1}{2}}. \quad (5.2)$$

Let q be as in Proposition 5.1. By (4.7), (4.12) and Propositions 5.1-5.2, we have

$$\begin{aligned} \mathcal{B}_1(v, q, \beta) &\ll q \sum_{\pm} \sum_{n|q} \sum_{m \geq 1} \frac{1}{nm} \sum_{n_1 | n} \sum_{n_2 | \frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \left| \Phi_{\beta}^{\pm} \left(\frac{mn^2}{q^3} \right) \right| \\ &\quad \times \sum_{\substack{-\frac{3q}{2} < b_j \leq \frac{3q}{2} \\ 1 \leq j \leq 3}} |\Psi(b_1, q, \beta)| |\Psi(b_2, q, \beta)| |\Psi(b_3, q, \beta)| |\mathcal{C}(b_1, b_2, b_3, n, m, v; q)| \\ &\ll_{\varepsilon} X^{\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2}} q_1^4 q_0^4 q_3'^{\frac{7}{2}} \sum_{\pm} \sum_{m \geq 1} \frac{1}{m} \sum_{n_1 | n} \sum_{n_2 | \frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \left| \Phi_{\beta}^{\pm} \left(\frac{mn^2}{q^3} \right) \right| \\ &\ll_{\varepsilon} X^{\varepsilon} (M + |\beta|^2 X^2) \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} q_1^4 q_0^4 q_3'^{\frac{7}{2}}. \end{aligned}$$

Then by (4.3), we have

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_1(v, q, \beta) d\beta \\
& \ll_{\varepsilon} X^{\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2} + \varepsilon} \sum_{q_1 | n} \sum_{\substack{q'_3 \leq Q/q_1 \\ q'_3 \text{ square-free}}} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^4 q_0^4 q_3'^{\frac{7}{2}} \int_{|\beta| \leq \frac{1}{q_1 q_0 q'_3 Q}} (M + |\beta|^2 X^2) d\beta \\
& \ll_{\varepsilon} X^{\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2} + \varepsilon} \sum_{q_1 | n} \sum_{q'_3 \leq Q/q_1} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^4 q_0^4 q_3'^{\frac{7}{2}} \left(\frac{M}{q_1 q_0 q'_3 Q} + \frac{X^2}{(q_1 q_0 q'_3 Q)^3} \right) \\
& \ll_{\varepsilon} \frac{X^{\varepsilon} M}{Q} \sum_{n \leq Q} n^{-\frac{3}{2} + \varepsilon} \sum_{q_1 | n} q_1^3 \sum_{q'_3 \leq Q/q_1} q_3'^{\frac{5}{2}} \left(\frac{Q}{q_1 q'_3} \right)^{\frac{7}{2}} \\
& \quad + \frac{X^{2+\varepsilon}}{Q^3} \sum_{n \leq Q} n^{-\frac{3}{2} + \varepsilon} \sum_{q_1 | n} q_1 \sum_{q'_3 \leq Q/q_1} q_3'^{\frac{1}{2}} \left(\frac{Q}{q_1 q'_3} \right)^{\frac{3}{2}} \\
& \ll_{\varepsilon} X^{\varepsilon} M Q^{\frac{5}{2}} + \frac{X^{2+\varepsilon}}{Q^{\frac{3}{2}}} \\
& \ll_{\varepsilon} M x^{\frac{5}{4} + \varepsilon}. \tag{5.3}
\end{aligned}$$

Moreover, by (4.7), (4.13) and Propositions 5.1-5.2, we have

$$\begin{aligned}
\mathcal{B}_2(v, q, \beta) & \ll_{\varepsilon} x^{\varepsilon} \left(\frac{x}{1 + |\beta|x} \right)^{\frac{1}{2}} \sum_{n \leq Q} n^{-\frac{3}{2}} q_1^3 q_0^3 q_3'^{\frac{5}{2}} X^{\varepsilon} n^{\varepsilon} (M + |\beta|^2 X^2) \\
& \ll_{\varepsilon} x^{\frac{1}{2} + \varepsilon} \left((1 + |\beta|x)^{\frac{3}{2}} + \frac{M}{\sqrt{1 + |\beta|x}} \right) \sum_{n \leq Q} n^{-\frac{3}{2} + \varepsilon} q_1^3 q_0^3 q_3'^{\frac{5}{2}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_2(v, q, \beta) d\beta \\
& \ll_{\varepsilon} x^{\frac{1}{2} + \varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2} + \varepsilon} \sum_{q_1 | n} \sum_{\substack{q'_3 \leq Q/q_1 \\ q'_3 \text{ square-free}}} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^3 q_0^3 q_3'^{\frac{5}{2}} \int_{|\beta| \leq \frac{1}{q_1 q_0 q'_3 Q}} (1 + |\beta|x)^{\frac{3}{2}} d\beta \\
& \quad + x^{\varepsilon} M \sum_{n \leq Q} n^{-\frac{3}{2} + \varepsilon} \sum_{q_1 | n} \sum_{\substack{q'_3 \leq Q/q_1 \\ q'_3 \text{ square-free}}} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^3 q_0^3 q_3'^{\frac{5}{2}} \int_{|\beta| \leq \frac{1}{q_1 q_0 q'_3 Q}} \frac{1}{\sqrt{x^{-1} + |\beta|}} d\beta
\end{aligned}$$

$$\begin{aligned}
& \ll_{\varepsilon} x^{\frac{1}{2}+\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^3 q_0^3 q'_3{}^{\frac{5}{2}} \left(\frac{1}{q_1 q_0 q'_3 Q} + \frac{x^{\frac{3}{2}}}{(q_1 q_0 q'_3 Q)^{\frac{5}{2}}} \right) \\
& + x^{\varepsilon} M \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^3 q_0^3 q'_3{}^{\frac{5}{2}} (q_1 q_0 q'_3 Q)^{-\frac{1}{2}} \\
& \ll_{\varepsilon} \frac{x^{\frac{1}{2}+\varepsilon}}{Q} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1^2 \sum_{q'_3 \leq Q/q_1} q'_3{}^{\frac{3}{2}} \left(\frac{Q}{q_1 q'_3} \right)^{\frac{5}{2}} \\
& + \frac{x^{2+\varepsilon}}{Q^{\frac{5}{2}}} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1^{\frac{1}{2}} \sum_{q'_3 \leq Q/q_1} \frac{Q}{q_1 q'_3} \\
& + \frac{x^{\varepsilon} M}{Q^{\frac{1}{2}}} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1^{\frac{5}{2}} \sum_{q'_3 \leq Q/q_1} q'_3{}^2 \left(\frac{Q}{q_1 q'_3} \right)^3 \\
& \ll_{\varepsilon} x^{\frac{1}{2}+\varepsilon} Q^{\frac{3}{2}} + \frac{x^{2+\varepsilon}}{Q^{\frac{3}{2}}} + x^{\varepsilon} M Q^{\frac{5}{2}} \\
& \ll_{\varepsilon} M x^{\frac{5}{4}+\varepsilon}.
\end{aligned} \tag{5.4}$$

Further, by (4.7), (4.14) and Propositions 5.1-5.2, we have

$$\begin{aligned}
\mathcal{B}_3(v, q, \beta) & \ll_{\varepsilon} \frac{x^{1+\varepsilon}}{1+|\beta|x} \sum_{n \leq Q} n^{-\frac{3}{2}} q_1^2 q_0^2 q'_3{}^{\frac{3}{2}} X^{\varepsilon} n^{\varepsilon} (M + |\beta|^2 X^2) \\
& \ll_{\varepsilon} x^{1+\varepsilon} \left(1 + |\beta|x + \frac{M}{1+|\beta|x} \right) \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} q_1^2 q_0^2 q'_3{}^{\frac{3}{2}}.
\end{aligned}$$

It follows from this estimate and (4.3) that

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_3(v, q, \beta) d\beta \\
& \ll_{\varepsilon} x^{1+\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{\substack{q'_3 \leq Q/q_1 \\ q'_3 \text{ square-free}}} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^2 q_0^2 q'_3{}^{\frac{3}{2}} \int_{|\beta| \leq \frac{1}{q_1 q_0 q'_3 Q}} (1 + |\beta|x) d\beta \\
& + x^{\varepsilon} M \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{\substack{q'_3 \leq Q/q_1 \\ q'_3 \text{ square-free}}} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^2 q_0^2 q'_3{}^{\frac{3}{2}} \int_{|\beta| \leq \frac{1}{q_1 q_0 q'_3 Q}} (x^{-1} + |\beta|)^{-1} d\beta
\end{aligned}$$

$$\begin{aligned}
&\ll_{\varepsilon} x^{1+\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^2 q_0^2 q'_3{}^{\frac{3}{2}} \left(\frac{1}{q_1 q_0 q'_3 Q} + \frac{x}{(q_1 q_0 q'_3 Q)^2} \right) \\
&\quad + x^{\varepsilon} M \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1^2 q_0^2 q'_3{}^{\frac{3}{2}} \\
&\ll_{\varepsilon} \frac{x^{1+\varepsilon}}{Q} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} q'_3{}^{\frac{1}{2}} \left(\frac{Q}{q_1 q'_3} \right)^{\frac{3}{2}} \\
&\quad + \frac{x^{2+\varepsilon}}{Q^2} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} q'_3{}^{-\frac{1}{2}} \left(\frac{Q}{q_1 q'_3} \right)^{\frac{1}{2}} \\
&\quad + x^{\varepsilon} M \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} q'_3{}^{\frac{3}{2}} \left(\frac{Q}{q_1 q'_3} \right)^{\frac{5}{2}} \\
&\ll_{\varepsilon} x^{1+\varepsilon} Q^{\frac{1}{2}} + \frac{x^{2+\varepsilon}}{Q^{\frac{3}{2}}} + x^{\varepsilon} M Q^{\frac{5}{2}} \\
&\ll_{\varepsilon} M x^{\frac{5}{4}+\varepsilon}.
\end{aligned} \tag{5.5}$$

Lastly, by (4.7), (4.15) and Propositions 5.1-5.2, we have

$$\begin{aligned}
\mathcal{B}_4(v, q, \beta) &\ll_{\varepsilon} x^{\varepsilon} \left(\frac{x}{1 + |\beta|x} \right)^{\frac{3}{2}} \sum_{n \leq Q} n^{-\frac{3}{2}} q_1 q_0 q'_3{}^{\frac{1}{2}} X^{\varepsilon} n^{\varepsilon} (M + |\beta|^2 X^2) \\
&\ll_{\varepsilon} \left(x^{\frac{3}{2}+\varepsilon} (1 + |\beta|x)^{\frac{1}{2}} + \frac{x^{\varepsilon} M}{(x^{-1} + |\beta|)^{\frac{3}{2}}} \right) \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} q_1 q_0 q'_3{}^{\frac{1}{2}}.
\end{aligned}$$

By (4.3) and the estimate as above, we have

$$\begin{aligned}
&\sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_4(v, q, \beta) d\beta \\
&\ll_{\varepsilon} x^{\frac{3}{2}+\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{\substack{q'_3 \leq Q/q_1 \\ q'_3 \text{ square-free}}} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1 q_0 q'_3{}^{\frac{1}{2}} \int_{|\beta| \leq \frac{1}{q_1 q_0 q'_3 Q}} (1 + |\beta|x)^{\frac{1}{2}} d\beta \\
&\quad + x^{\varepsilon} M \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{\substack{q'_3 \leq Q/q_1 \\ q'_3 \text{ square-free}}} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1 q_0 q'_3{}^{\frac{1}{2}} \int_{|\beta| \leq \frac{1}{q_1 q_0 q'_3 Q}} (x^{-1} + |\beta|)^{-\frac{3}{2}} d\beta
\end{aligned}$$

$$\begin{aligned}
& \ll_{\varepsilon} x^{\frac{3}{2}+\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1 q_0 q'_3{}^{\frac{1}{2}} \left(\frac{1}{q_1 q_0 q'_3 Q} + \frac{x^{\frac{1}{2}}}{(q_1 q_0 q'_3 Q)^{\frac{3}{2}}} \right) \\
& + M x^{\frac{1}{2}+\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} \sum_{\substack{q_0 \leq Q/(q_1 q'_3) \\ 4q_0 \text{ square-full}}} q_1 q_0 q'_3{}^{\frac{1}{2}} \\
& \ll_{\varepsilon} \frac{x^{\frac{3}{2}+\varepsilon}}{Q} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} \sum_{q'_3 \leq Q/q_1} q'_3{}^{-\frac{1}{2}} \left(\frac{Q}{q_1 q'_3} \right)^{\frac{1}{2}} \\
& + \frac{x^{2+\varepsilon}}{Q^{\frac{3}{2}}} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1^{-\frac{1}{2}} \sum_{q'_3 \leq Q/q_1} q'_3{}^{-1} \\
& + M x^{\frac{1}{2}+\varepsilon} \sum_{n \leq Q} n^{-\frac{3}{2}+\varepsilon} \sum_{q_1|n} q_1 \sum_{q'_3 \leq Q/q_1} q'_3{}^{\frac{1}{2}} \left(\frac{Q}{q_1 q'_3} \right)^{\frac{3}{2}} \\
& \ll_{\varepsilon} \frac{x^{\frac{3}{2}+\varepsilon}}{Q^{\frac{1}{2}}} + \frac{x^{2+\varepsilon}}{Q^{\frac{3}{2}}} + M x^{\frac{1}{2}+\varepsilon} Q^{\frac{3}{2}} \\
& \ll_{\varepsilon} M x^{\frac{5}{4}+\varepsilon}. \tag{5.6}
\end{aligned}$$

By (4.2), (4.11) and (5.3)-(5.6), the contribution from \mathcal{B}_j , $j = 1, 2, 3, 4$, is $O_{\varepsilon}(M x^{\frac{5}{4}+\varepsilon})$.

6. CONTRIBUTION OF \mathcal{B}_j , $5 \leq j \leq 13$

First, we note that (recall (3.2) and (3.3))

$$P_j(n, q) \ll (\log(n+2)(q+2))^j, \quad j = 1, 2 \tag{6.1}$$

and

$$\widetilde{\phi}_{\beta}^{(j)}(1) = \int_0^{\infty} \phi\left(\frac{u}{X}\right) e(-\beta u) (\log u)^j du \ll \frac{x(\log x)^j}{1 + |\beta|x}, \quad j = 0, 1, 2. \tag{6.2}$$

Next, bounding the character sum $\mathcal{C}(b_1, b_2, b_3, n, m, v; q)$ by Weil's bound for Kloosterman sums, we have

$$\mathcal{C}(b_1, b_2, b_3, n, m, v; q) \ll q^{\frac{5}{2}} \left(\frac{q}{n}\right)^{\frac{1}{2}} \tau\left(\frac{q}{n}\right) \ll_{\varepsilon} q^{3+\varepsilon} n^{-\frac{1}{2}}. \tag{6.3}$$

By (4.7), (4.16)-(4.18) and (6.1)-(6.3), we have, for $j = 5, 6, 7$,

$$\mathcal{B}_j(v, q, \beta) \ll_{\varepsilon} \frac{1}{q^2} \frac{x(\log x)^2}{1 + |\beta|x} \sum_{n|q} n \tau(n) q^{3+\varepsilon} n^{-\frac{1}{2}} (\log(q+2))^5 \ll_{\varepsilon} \frac{x^{\varepsilon} q^{\frac{3}{2}+\varepsilon}}{x^{-1} + |\beta|}. \tag{6.4}$$

By (4.3) and (6.4), we obtain, for $j = 5, 6, 7$,

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_j(v, q, \beta) d\beta \\
& \ll_{\varepsilon} x^{\varepsilon} \sum_{q \leq Q} q^{\frac{3}{2} + \varepsilon} \int_{|\beta| \leq \frac{1}{qQ}} \frac{1}{x^{-1} + |\beta|} d\beta \\
& \ll_{\varepsilon} x^{\varepsilon} Q^{\frac{5}{2} + \varepsilon}.
\end{aligned} \tag{6.5}$$

By (4.7), (4.19)-(4.21) and (6.1)-(6.3), we have, for $j = 8, 9, 10$,

$$\begin{aligned}
\mathcal{B}_j(v, q, \beta) & \ll_{\varepsilon} \frac{1}{q^3} \frac{x(\log x)^2}{1 + |\beta|x} \left(\frac{x}{1 + |\beta|x} \right)^{\frac{1}{2}} \sum_{n|q} n \tau(n) q^{3+\varepsilon} n^{-\frac{1}{2}} (\log(q+2))^4 \\
& \ll_{\varepsilon} x^{\varepsilon} q^{\frac{1}{2} + \varepsilon} \left(\frac{1}{x^{-1} + |\beta|} \right)^{\frac{3}{2}}.
\end{aligned} \tag{6.6}$$

By (4.3) and (6.6), we obtain, for $j = 8, 9, 10$,

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_j(v, q, \beta) d\beta \\
& \ll_{\varepsilon} x^{\varepsilon} \sum_{q \leq Q} q^{\frac{1}{2} + \varepsilon} \int_{|\beta| \leq \frac{1}{qQ}} \left(\frac{1}{x^{-1} + |\beta|} \right)^{\frac{3}{2}} d\beta \\
& \ll_{\varepsilon} x^{\frac{1}{2} + \varepsilon} Q^{\frac{3}{2} + \varepsilon}.
\end{aligned} \tag{6.7}$$

By (4.7), (4.22)-(4.24) and (6.1)-(6.3), we have, for $j = 11, 12, 13$,

$$\begin{aligned}
\mathcal{B}_j(v, q, \beta) & \ll_{\varepsilon} \frac{1}{q^4} \frac{x(\log x)^2}{1 + |\beta|x} \frac{x}{1 + |\beta|x} \sum_{n|q} n \tau(n) q^{3+\varepsilon} n^{-\frac{1}{2}} (\log(q+2))^3 \\
& \ll_{\varepsilon} x^{\varepsilon} q^{-\frac{1}{2} + \varepsilon} \left(\frac{1}{x^{-1} + |\beta|} \right)^2.
\end{aligned} \tag{6.8}$$

By (4.3) and (6.8), we obtain, for $j = 11, 12, 13$,

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_j(v, q, \beta) d\beta \\
& \ll_{\varepsilon} x^{\varepsilon} \sum_{q \leq Q} q^{-\frac{1}{2} + \varepsilon} \int_{|\beta| \leq \frac{1}{qQ}} \left(\frac{1}{x^{-1} + |\beta|} \right)^2 d\beta \\
& \ll_{\varepsilon} x^{1+\varepsilon} Q^{\frac{1}{2} + \varepsilon}.
\end{aligned} \tag{6.9}$$

By (4.2), (4.11), (6.5), (6.7) and (6.9), the contribution from \mathcal{B}_j , $5 \leq j \leq 13$, is $O_\varepsilon(x^{\frac{5}{4}+\varepsilon})$.

7. COMPUTATION OF THE MAIN TERMS

The three sums $\mathcal{B}_{14}(v, q, \beta)$, $\mathcal{B}_{15}(v, q, \beta)$ and $\mathcal{B}_{16}(v, q, \beta)$ in (4.25)-(4.27) contribute the main terms. Replacing $\widetilde{\phi}_\beta^{(j)}(1)$ by

$$\vartheta^{b,j}(\beta) = \int_{X/2}^X e(-\beta u)(\log u)^j du, \quad j = 0, 1, 2, \quad (7.1)$$

we need to estimate the remainder terms from

$$\vartheta^{\sharp,j}(\beta) = \widetilde{\phi}_\beta^{(j)}(1) - \vartheta^{b,j}(\beta).$$

Write correspondingly

$$\mathcal{B}_j^\sharp(v, q, \beta) = \mathcal{B}_j(v, q, \beta) - \mathcal{B}_j^b(v, q, \beta), \quad j = 14, 15, 16.$$

First, we evaluate the remainder terms from $\mathcal{B}_j^\sharp(v, q, \beta)$, $j = 14, 15, 16$. Notice that

$$\vartheta^{\sharp,j}(\beta) = \int_{X/2}^X \left(\phi\left(\frac{u}{X}\right) - 1 \right) e(-\beta u)(\log u)^j du \ll XM^{-1}(\log X)^j.$$

Hence

$$\begin{aligned} & \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{14}^\sharp(v, q, \beta) d\beta \\ &= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \vartheta^{\sharp,0}(\beta) \Psi_0(\beta)^3 \\ & \quad \times \sum_{n|q} n \tau(n) P_2(n, q) \mathcal{C}(0, 0, 0, n, 0, v; q) d\beta \\ &\ll_\varepsilon X^{1+\varepsilon} M^{-1} \sum_{q \leq Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}} \tau(n) |P_2(n, q)| \int_{|\beta| \leq \frac{1}{qQ}} \left(\frac{1}{x^{-1} + |\beta|} \right)^{\frac{3}{2}} d\beta \\ &\ll_\varepsilon x^{\frac{3}{2}+\varepsilon} M^{-1} \sum_{q \leq Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}} \tau(n) |P_2(n, q)| \\ &\ll_\varepsilon x^{\frac{3}{2}+\varepsilon} M^{-1}. \end{aligned} \quad (7.2)$$

Here we have used (4.3), (5.2) and (6.3). Similarly,

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{15}^\sharp(v, q, \beta) d\beta \\
&= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \vartheta^{\sharp, 1}(\beta) \Psi_0(\beta)^3 \\
&\quad \times \sum_{n|q} n \tau(n) P_1(n, q) \mathcal{C}(0, 0, 0, n, 0, v; q) d\beta \\
&\ll_{\varepsilon} X^{1+\varepsilon} M^{-1} \sum_{q \leq Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}} \tau(n) |P_1(n, q)| \int_{|\beta| \leq \frac{1}{qQ}} \left(\frac{1}{x^{-1} + |\beta|} \right)^{\frac{3}{2}} d\beta \\
&\ll_{\varepsilon} x^{\frac{3}{2}+\varepsilon} M^{-1} \sum_{q \leq Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}} \tau(n) |P_1(n, q)| \\
&\ll_{\varepsilon} x^{\frac{3}{2}+\varepsilon} M^{-1}, \tag{7.3}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{16}^\sharp(v, q, \beta) d\beta \\
&= \frac{1}{4} \sum_{q \leq Q} \frac{1}{q^5} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \vartheta^{\sharp, 2}(\beta) \Psi_0(\beta)^3 \sum_{n|q} n \tau(n) \mathcal{C}(0, 0, 0, n, 0, v; q) d\beta \\
&\ll_{\varepsilon} X^{1+\varepsilon} M^{-1} \sum_{q \leq Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}} \tau(n) \int_{|\beta| \leq \frac{1}{qQ}} \left(\frac{1}{x^{-1} + |\beta|} \right)^{\frac{3}{2}} d\beta \\
&\ll_{\varepsilon} x^{\frac{3}{2}+\varepsilon} M^{-1} \sum_{q \leq Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}} \tau(n) \\
&\ll_{\varepsilon} x^{\frac{3}{2}+\varepsilon} M^{-1}. \tag{7.4}
\end{aligned}$$

Next, we want to compute the contributions from $\mathcal{B}_j^b(v, q, \beta)$ which constitute the main terms. Interchanging the order of summation over a and the integration over β , we have

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{14}^b(v, q, \beta) d\beta \\
&= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \vartheta^{b,0}(\beta) \Psi_0(\beta)^3 \sum_{n|q} n \tau(n) P_2(n, q) \\
&\quad \times \sum_{a=1}^q e\left(\frac{-\bar{a}v}{q}\right) G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right) d\beta \\
&= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \sum_{a=1}^q G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right) \int_{\mathcal{M}(a, q)} \vartheta^{b,0}(\beta) \Psi_0(\beta)^3 d\beta.
\end{aligned}$$

Note that

$$\left[-\frac{1}{2qQ}, \frac{1}{2qQ}\right] \subseteq \mathcal{M}(a, q) \subseteq \left[-\frac{1}{qQ}, \frac{1}{qQ}\right].$$

As in [16], we write $\mathcal{M}(a, q)$ as

$$\mathcal{M}(a, q) = \mathcal{M}(a, q) \setminus \left[-\frac{1}{2qQ}, \frac{1}{2qQ}\right] \cup \left[-\frac{1}{2qQ}, \frac{1}{2qQ}\right].$$

Accordingly,

$$\begin{aligned}
& \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{14}^b(v, q, \beta) d\beta \\
&= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \sum_{a=1}^q G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right) \int_{-\infty}^{\infty} \vartheta^{b,0}(\beta) \Psi_0(\beta)^3 d\beta \\
&\quad + \mathcal{B}_{14}^* - \mathcal{B}_{14}^{**}, \tag{7.5}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}_{14}^* &= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \sum_{a=1}^q G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right) \\
&\quad \times \int_{\mathcal{M}(a, q) \setminus \left[-\frac{1}{2qQ}, \frac{1}{2qQ}\right]} \vartheta^{b,0}(\beta) \Psi_0(\beta)^3 d\beta \\
\mathcal{B}_{14}^{**} &= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \sum_{a=1}^q G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right) \int_{|\beta| > \frac{1}{2qQ}} \vartheta^{b,0}(\beta) \Psi_0(\beta)^3 d\beta.
\end{aligned}$$

Interchanging the order of summation over a and the integration over β again, we have

$$\begin{aligned}
\mathcal{B}_{14}^* &= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \int_{\frac{1}{2qQ} \leq |\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \vartheta^{b,0}(\beta) \Psi_0(\beta)^3 \\
&\quad \sum_{a=1}^q e\left(\frac{-\bar{a}v}{q}\right) G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right) d\beta \\
&\ll_{\varepsilon} \sum_{q \leq Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}} \tau(n) |P_2(n, q)| \int_{\frac{1}{2qQ} \leq |\beta| \leq \frac{1}{qQ}} |\beta|^{-\frac{5}{2}} d\beta \\
&\ll_{\varepsilon} Q^{\frac{5}{2}+\varepsilon},
\end{aligned} \tag{7.6}$$

and

$$\begin{aligned}
\mathcal{B}_{14}^{**} &= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \int_{|\beta| > \frac{1}{2qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \vartheta^{b,0}(\beta) \Psi_0(\beta)^3 \\
&\quad \sum_{a=1}^q e\left(\frac{-\bar{a}v}{q}\right) G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right) d\beta \\
&\ll_{\varepsilon} \sum_{q \leq Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}} \tau(n) |P_2(n, q)| \int_{|\beta| > \frac{1}{2qQ}} |\beta|^{-\frac{5}{2}} d\beta \\
&\ll_{\varepsilon} Q^{\frac{5}{2}+\varepsilon}.
\end{aligned} \tag{7.7}$$

Here we have used (4.3), (5.2) and (6.1)-(6.3). By (7.5)-(7.7), we obtain

$$\begin{aligned}
&\sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{14}^b(v, q, \beta) d\beta \\
&= \frac{1}{2} \sum_{q \leq Q} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \sum_{a=1}^q G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right) \\
&\quad \times \int_{-\infty}^{\infty} \vartheta^{b,0}(\beta) \Psi_0(\beta)^3 d\beta + O_{\varepsilon}\left(x^{\frac{5}{4}+\varepsilon}\right).
\end{aligned} \tag{7.8}$$

Moreover, by (4.6),

$$\Psi_0(\beta) = \int_0^{\sqrt{x}} e(\beta v^2) dv = x^{\frac{1}{2}} \int_0^1 e(\beta x v^2) dv.$$

It follows that

$$\int_{-\infty}^{\infty} \vartheta^{b,0}(\beta) \Psi_0(\beta)^3 d\beta = x^{\frac{3}{2}} \int_{-\infty}^{\infty} \left(\int_{X/2}^X e(-\beta u) du \right) \left(\int_0^1 e(\beta x v^2) dv \right)^3 d\beta := x^{\frac{3}{2}} \mathcal{I}_0(X),$$

where

$$\mathcal{I}_0(X) = \int_{-\infty}^{\infty} \left(\int_{X/2}^X e(-\beta u) du \right) \left(\int_0^1 e(\beta x v^2) dv \right)^3 d\beta. \quad (7.9)$$

Substituting in (7.8) and by (7.2), we have

$$\begin{aligned} & \sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{14}(v, q, \beta) d\beta \\ &= \frac{1}{2} \mathcal{I}_0(X) x^{\frac{3}{2}} \sum_{q \leq Q} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \mathcal{C}(0, 0, 0, n, 0, 0; q) + O_{\varepsilon} \left(x^{\frac{5}{4}+\varepsilon} + x^{\frac{3}{2}+\varepsilon} M^{-1} \right). \end{aligned} \quad (7.10)$$

Further, by (6.1) and (6.3), we have

$$\begin{aligned} & \sum_{q > Q} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \mathcal{C}(0, 0, 0, n, 0, 0; q) \\ & \ll_{\varepsilon} \sum_{q > Q} q^{-2+\varepsilon} \sum_{n|q} n^{\frac{1}{2}} \tau(n) |P_2(n, q)| \\ & \ll_{\varepsilon} Q^{-\frac{1}{2}+\varepsilon}. \end{aligned} \quad (7.11)$$

By (7.10) and (7.11), we conclude that

$$\sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{14}(v, q, \beta) d\beta = \frac{1}{2} \mathcal{I}_0(X) \mathcal{C}_2 x^{\frac{3}{2}} + O_{\varepsilon} \left(x^{\frac{5}{4}+\varepsilon} + x^{\frac{3}{2}+\varepsilon} M^{-1} \right), \quad (7.12)$$

where

$$\mathcal{C}_2 = \sum_{q=1}^{\infty} \frac{1}{q^5} \sum_{n|q} n \tau(n) P_2(n, q) \sum_{a=1}^q {}^* G(a, 0; q)^3 S \left(-\bar{a}, 0; \frac{q}{n} \right). \quad (7.13)$$

Similarly, we have

$$\sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{15}^b(v, q, \beta) d\beta = \frac{1}{2} \mathcal{I}_1(X) \mathcal{C}_1 x^{\frac{3}{2}} + O_{\varepsilon} \left(x^{\frac{5}{4}+\varepsilon} \right), \quad (7.14)$$

where

$$\mathcal{I}_1(X) = \int_{-\infty}^{\infty} \left(\int_{X/2}^X e(-\beta u) (\log u) du \right) \left(\int_0^1 e(\beta x v^2) dv \right)^3 d\beta, \quad (7.15)$$

and

$$\mathcal{C}_1 = \sum_{q=1}^{\infty} \frac{1}{q^5} \sum_{n|q} n\tau(n)P_1(n, q) \sum_{a=1}^q{}^* G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right). \quad (7.16)$$

By (7.3) and (7.14), we have

$$\sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{15}(v, q, \beta) d\beta = \frac{1}{2} \mathcal{I}_1(X) \mathcal{C}_1 x^{\frac{3}{2}} + O_{\varepsilon}\left(x^{\frac{5}{4}+\varepsilon} + x^{\frac{3}{2}+\varepsilon} M^{-1}\right). \quad (7.17)$$

Finally, we have

$$\sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{16}^b(v, q, \beta) d\beta = \frac{1}{4} \mathcal{I}_2(X) \mathcal{C}_0 x^{\frac{3}{2}} + O_{\varepsilon}\left(x^{\frac{5}{4}+\varepsilon}\right), \quad (7.18)$$

where

$$\mathcal{I}_2(X) = \int_{-\infty}^{\infty} \left(\int_{X/2}^X e(-\beta u) (\log u)^2 du \right) \left(\int_0^1 e(\beta x v^2) dv \right)^3 d\beta, \quad (7.19)$$

and

$$\mathcal{C}_0 = \sum_{q=1}^{\infty} \frac{1}{q^5} \sum_{n|q} n\tau(n) \sum_{a=1}^q{}^* G(a, 0; q)^3 S\left(-\bar{a}, 0; \frac{q}{n}\right). \quad (7.20)$$

By (7.4) and (7.18), we obtain

$$\sum_{q \leq Q} \int_{|\beta| \leq \frac{1}{qQ}} \sum_{v \bmod q} \varrho(v, q, \beta) \mathcal{B}_{16}(v, q, \beta) d\beta = \frac{1}{4} \mathcal{I}_2(X) \mathcal{C}_0 X^{\frac{3}{2}} + O_{\varepsilon}\left(x^{\frac{5}{4}+\varepsilon} + x^{\frac{3}{2}+\varepsilon} M^{-1}\right) \quad (7.21)$$

By (4.2), (4.11), (7.12), (7.17) and (7.21), the contribution from \mathcal{B}_j , $14 \leq j \leq 16$, is

$$\frac{1}{2} \mathcal{I}_0(X) \mathcal{C}_2 x^{\frac{3}{2}} + \frac{1}{2} \mathcal{I}_1(X) \mathcal{C}_1 x^{\frac{3}{2}} + \frac{1}{4} \mathcal{I}_2(X) \mathcal{C}_0 x^{\frac{3}{2}} + O_{\varepsilon}\left(x^{\frac{5}{4}+\varepsilon} + x^{\frac{3}{2}+\varepsilon} M^{-1}\right),$$

where \mathcal{I}_j ($0 \leq j \leq 2$) and \mathcal{C}_j ($0 \leq j \leq 2$) are defined in (7.9), (7.15), (7.19) and (7.13), (7.16), (7.20), respectively.

8. PROOF OF PROPOSITION 5.2

Recall $\Phi_{\beta}^{\pm}(y)$ in (4.9) which we relabel as

$$\Phi_{\beta}^{\pm}(y) = \Phi_0(y, \beta) \pm \frac{1}{i\pi^3 y} \Phi_1(y, \beta), \quad (8.1)$$

where for $\sigma > -1 - k$,

$$\Phi_k(y, \beta) = (\pi^3 y)^k \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=\sigma} (\pi^3 y)^{-s} \frac{\Gamma\left(\frac{1+s+k}{2}\right)^3}{\Gamma\left(\frac{-s+k}{2}\right)^3} \widetilde{\phi}_{\beta}(-s) ds \quad (8.2)$$

with $\phi_\beta(y) = \phi\left(\frac{y}{X}\right) e(-\beta y)$. Note that

$$\phi_\beta^{(j)}(y) \ll_j \left(\frac{M + |\beta|X}{X} \right)^j.$$

By Lemma 3.2, we have

$$\begin{aligned} & \sum_{\pm} \sum_{m \geq 1} \frac{1}{m} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \left| \Phi_\beta^\pm \left(\frac{mn^2}{q^3} \right) \right| \\ &= \sum_{\pm} \sum_{\frac{mn^2}{q^3} X < X^\varepsilon (M + |\beta|X)^3} \frac{1}{m} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \left| \Phi_\beta^\pm \left(\frac{mn^2}{q^3} \right) \right| + O_\varepsilon(1). \end{aligned} \quad (8.3)$$

Moreover, by (3.1), trivially, we have

$$\sum_{m \leq L} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \ll \sum_{m \leq L} \tau_3(n) \tau_3(m) \ll_\varepsilon n^\varepsilon L^{1+\varepsilon}. \quad (8.4)$$

For $yX \ll X^\varepsilon$, we move the line of integration in (8.2) to $\sigma = -1 + \varepsilon$ to obtain

$$\begin{aligned} \Phi_k(y, \beta) &= (\pi^3 y)^k \frac{1}{2\pi i} \int_{\text{Re}(s)=-1+\varepsilon} (\pi^3 y)^{-s} \frac{\Gamma\left(\frac{1+s+k}{2}\right)^3}{\Gamma\left(\frac{-s+k}{2}\right)^3} \widetilde{\phi}_\beta(-s) ds \\ &\ll y^k (yX)^{1-\varepsilon} \int_{-\infty}^{\infty} (1+|t|)^{-\frac{3}{2}+3\varepsilon} dt \\ &\ll_\varepsilon y^k X^\varepsilon. \end{aligned} \quad (8.5)$$

Here we have used Stirling's formula and the estimate $\widetilde{\phi}_\beta(-s) = \int_0^\infty \phi_\beta(u) u^{-s-1} du \ll X^{-\sigma}$. By (8.1), (8.4) and (8.5), we have

$$\begin{aligned} & \sum_{\pm} \sum_{\frac{mn^2}{q^3} X \ll X^\varepsilon} \frac{1}{m} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \left| \Phi_\beta^\pm \left(\frac{mn^2}{q^3} \right) \right| \\ &\ll_\varepsilon X^\varepsilon \max_{1 \leq L \ll \frac{q^3 X^\varepsilon}{n^2 X}} \frac{1}{L} \sum_{L < m \leq 2L} \sum_{n_1|n} \sum_{n_2|\frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \\ &\ll_\varepsilon X^\varepsilon \max_{1 \leq L \ll \frac{q^3 X^\varepsilon}{n^2 X}} \frac{1}{L} n^\varepsilon L^{1+\varepsilon} \\ &\ll_\varepsilon X^\varepsilon n^\varepsilon. \end{aligned} \quad (8.6)$$

For $yX > X^\varepsilon$, by Lemma 3.3, we have

$$\begin{aligned}\Phi_k(y, \beta) &= (\pi^3 y)^{k+1} \sum_{j=1}^{\ell} \int_0^{\infty} \phi\left(\frac{u}{X}\right) e(-\beta u) \left(a_k(j) e\left(3(yu)^{\frac{1}{3}}\right) + b_k(j) e\left(-3(yu)^{\frac{1}{3}}\right) \right) \\ &\quad \times \frac{du}{(\pi^3 y u)^{\frac{1}{3}}} + O_{\varepsilon, \ell} \left((\pi^3 y)^k (\pi^3 y X)^{-\frac{\ell}{3} + \frac{1}{2} + \varepsilon} \right),\end{aligned}$$

where $a_k(j)$ and $b_k(j)$ are constants. Then by (8.1),

$$\Phi_{\beta}^{\pm}(y) \ll_{\varepsilon, \ell} y \sum_{j=1}^{\ell} y^{-\frac{j}{3}} (|\mathcal{I}_j(y, \beta)| + |\mathcal{J}_j(y, \beta)|) + (yX)^{-\frac{\ell}{3} + \frac{1}{2} + \varepsilon}, \quad (8.7)$$

with

$$\begin{aligned}\mathcal{I}_j(y, \beta) &= \int_0^{\infty} u^{-\frac{j}{3}} \phi\left(\frac{u}{X}\right) e(-\beta u) e\left(3(yu)^{\frac{1}{3}}\right) du, \\ \mathcal{J}_j(y, \beta) &= \int_0^{\infty} u^{-\frac{j}{3}} \phi\left(\frac{u}{X}\right) e(-\beta u) e\left(-3(yu)^{\frac{1}{3}}\right) du.\end{aligned}$$

By partial integration twice, we have

$$\mathcal{I}_j(y, \beta), \mathcal{J}_j(y, \beta) \ll (yX)^{-\frac{2}{3}} X^{1-\frac{j}{3}} (M + |\beta|^2 X^2). \quad (8.8)$$

Taking $\ell = 3$. By (8.7) and (8.8), we have

$$\Phi_{\beta}^{\pm}(y) \ll_{\varepsilon} (yX)^{\frac{1}{3}} (M + |\beta|^2 X^2) \sum_{j=1}^3 (yX)^{-\frac{j}{3}} + (yX)^{-\frac{1}{2} + \varepsilon} \ll_{\varepsilon} M + |\beta|^2 X^2. \quad (8.9)$$

By (8.4) and (8.9), we have

$$\begin{aligned}& \sum_{\pm} \sum_{X^{\varepsilon} < \frac{n^2 m}{q^3} X < X^{\varepsilon} (M + |\beta| X)^3} \frac{1}{m} \sum_{n_1 | n} \sum_{n_2 | \frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \left| \Phi_{\beta}^{\pm} \left(\frac{mn^2}{q^3} \right) \right| \\ & \ll_{\varepsilon} (M + |\beta|^2 X^2) (\log X) \max_{\frac{q^3 X^{\varepsilon}}{n^2 X} < L < \frac{q^3 X^{\varepsilon} (M + |\beta| X)^3}{n^2 X}} \frac{1}{L} \sum_{L < m \leq 2L} \sum_{n_1 | n} \sum_{n_2 | \frac{n}{n_1}} \sigma_{0,0} \left(\frac{n}{n_1 n_2}, m \right) \\ & \ll_{\varepsilon} X^{\varepsilon} n^{\varepsilon} (M + |\beta|^2 X^2).\end{aligned} \quad (8.10)$$

Then Proposition 5.2 follows from (8.3), (8.6) and (8.10).

9. ESTIMATION OF THE CHARACTER SUM $\mathcal{C}(b_1, b_2, b_3, n, m, v; q)$

In this section, we shall prove Proposition 5.1. Let $b_1, b_2, b_3, n, m, v \in \mathbb{Z}$ and $n|q$. We recall $\mathcal{C}(b_1, b_2, b_3, n, m, v; q)$ in (4.28) which we relabel as

$$\mathcal{C}(b_1, b_2, b_3, n, m, v; q) = \sum_{a \bmod q}^* e\left(\frac{-\bar{a}v}{q}\right) G(a, b_1; q) G(a, b_2; q) G(a, b_3; q) S\left(-\bar{a}, m; \frac{q}{n}\right) \quad (9.1)$$

where $G(a, b; q)$ is the Gauss sum defined in (4.5). Here for notation simplicity, we have replaced $\pm m$ in (4.28) by m , which does not affect our argument.

We first list some well-known results for $G(a, b; q)$ (see for example Lemma 5.4.5 in [7]). For $(a, q) = 1$, we have

$$G(a, b; q) \ll \sqrt{q}. \quad (9.2)$$

For $(2a, q) = 1$, we have

$$G(a, b; q) = e\left(-\frac{4\bar{a}b^2}{q}\right) G(a, 0; q) \quad (9.3)$$

and

$$G(a, 0; q) = \left(\frac{a}{q}\right) \epsilon_q \sqrt{q}, \quad (9.4)$$

where $\epsilon_q = \begin{cases} 1, & \text{if } q \equiv 1 \pmod{4}, \\ i, & \text{if } q \equiv -1 \pmod{4}. \end{cases}$

As in [12], we factor q as $q = q_1 q_2 q_3$, where q_1 is the largest factor of q such that $q_1 | n$, $(q_1, q_2 q_3) = 1$, $q_2 | n^\infty$, $(q_2, q_3) = 1$. Then q_2 is square-full and $n | q_1 q_2$. Denote temporarily $q' = q_1 q_2$ and $\hat{q} = \frac{q'}{n}$. Then we have

$$\begin{aligned} & \mathcal{C}(b_1, b_2, b_3, n, m, v; q) \\ &= \sum_{a_1 \bmod q'}^* e\left(\frac{-\bar{a}_1 v}{q'}\right) G(a_1, b_1; q') G(a_1, b_2; q') G(a_1, b_3; q') S(-\bar{a}_1 q_3, m \bar{q}_3^{-2}; \hat{q}) \\ & \quad \times \sum_{a_2 \bmod q_3}^* e\left(\frac{-\bar{a}_2 v}{q_3}\right) G(a_2, b_1; q_3) G(a_2, b_2; q_3) G(a_2, b_3; q_3) S(-\bar{a}_2 q', m \bar{q}'^2; q_3) \\ &:= \mathcal{C}^*(b_1, b_2, b_3, n, m, v; q') \mathcal{C}^{**}(b_1, b_2, b_3, n, m, v; q_3) \end{aligned} \quad (9.5)$$

say.

By (9.2) and Weil's bound for Kloosterman sum we have

$$\mathcal{C}^*(b_1, b_2, b_3, n, m, v; q') \ll q'^{\frac{5}{2}} \left(\bar{a}_1 q_3, m \bar{q}_3^{-2}, \frac{q'}{n}\right)^{\frac{1}{2}} \left(\frac{q'}{n}\right)^{\frac{1}{2}} \tau\left(\frac{q'}{n}\right) \ll \frac{q_1^3 q_2^3 \tau(q_1 q_2)}{\sqrt{n}}. \quad (9.6)$$

To estimate $\mathcal{C}^{**}(b_1, b_2, b_3, n, m, v; q_3)$, we further factor q_3 as $q_3 = q'_3 q''_3$ with $(q'_3, 2q''_3) = 1$, q'_3 square-free and $4q''_3$ square-full. Then

$$\mathcal{C}^{**}(b_1, b_2, b_3, n, m, v; q_3) = \mathcal{C}_1^{**}(b_1, b_2, b_3, n, m, v; q'_3) \mathcal{C}_2^{**}(b_1, b_2, b_3, n, m, v; q''_3), \quad (9.7)$$

where

$$\begin{aligned} \mathcal{C}_1^{**}(b_1, b_2, b_3, n, m, v; q'_3) &= \sum_{\gamma \bmod q'_3}^* e\left(\frac{-\bar{\gamma} v}{q'_3}\right) G(\gamma, b_1; q'_3) G(\gamma, b_2; q'_3) G(\gamma, b_3; q'_3) \\ & \quad \times S(-\bar{\gamma} q''_3 q', m \bar{q}'^2 \bar{q}_3^{-2}; q'_3), \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_2^{**}(b_1, b_2, b_3, n, m, v; q_3'') &= \sum_{\gamma \bmod q_3''}^* e\left(\frac{-\bar{\gamma}v}{q_3''}\right) G(\gamma, b_1; q_3'') G(\gamma, b_2; q_3'') G(\gamma, b_3; q_3'') \\ &\quad \times S(-\bar{\gamma}q_3'q', m\bar{q}^2\bar{q}_3''^2; q_3''). \end{aligned}$$

We estimate $\mathcal{C}_2^{**}(b_1, b_2, b_3, n, m, v; q_3'')$ similarly as $\mathcal{C}^*(b_1, b_2, b_3, n, m, v; q')$ getting

$$\mathcal{C}_2^{**}(b_1, b_2, b_3, n, m, v; q_3'') \ll q_3''^3 \tau(q_3''). \quad (9.8)$$

To estimate $\mathcal{C}_1^{**} := \mathcal{C}_1^{**}(b_1, b_2, b_3, n, m, v; q_3')$, we factor q_3' as $q_3' = p_1 p_2 \cdots p_s$, p_i prime, and correspondingly,

$$\mathcal{C}_1^{**} = \prod_{i=1}^s \mathcal{T}(b_1, b_2, b_3, q'_i q_3'' p'_i, m\bar{q}^2 \bar{q}_3''^2 p_i'^2; p_i), \quad (9.9)$$

where $p'_i = q'_i/p_i$ and

$$\mathcal{T}(b_1, b_2, b_3, r_1, r_2 m; p) = \sum_{z \bmod p}^* e\left(\frac{-v\bar{z}}{p}\right) G(z, b_1; p) G(z, b_2; p) G(z, b_3; p) S(-r_1 \bar{z}, r_2 m; p)$$

with $(p, 2r_1 r_2) = 1$.

By (9.3) and (9.4), we write

$$\mathcal{T}(b_1, b_2, b_3, r_1, r_2 m; p) = \epsilon_p^3 p^{\frac{3}{2}} \sum_{z \bmod p}^* \left(\frac{z}{p}\right) e\left(\frac{-\bar{4}(4v + b_1^2 + b_2^2 + b_3^2)\bar{z}}{p}\right) S(-r_1 \bar{z}, r_2 m; p).$$

By (9.5)-(9.9), Proposition 5.1 follows from the following lemma.

Lemma 9.1. *We have*

$$\mathcal{T}(b_1, b_2, b_3, r_1, r_2 m; p) \ll p^{\frac{5}{2}}.$$

Proof. If $p|m$, then $S(-r_1 \bar{z}, r_2 m; p) = -1$ and trivially,

$$\mathcal{T}(b_1, b_2, b_3, r_1, r_2 m; p) = -\epsilon_p^3 p^{\frac{3}{2}} \sum_{z \bmod p}^* \left(\frac{z}{p}\right) e\left(\frac{-\bar{4}(4v + b_1^2 + b_2^2 + b_3^2)\bar{z}}{p}\right) \ll p^{\frac{5}{2}}.$$

If $p \nmid m$, we open the Kloosterman sum to obtain

$$\mathcal{T}(b_1, b_2, b_3, r_1, r_2 m; p) = \epsilon_p^3 p^{\frac{3}{2}} \sum_{y, z \in \mathbb{F}_p^\times} \left(\frac{z}{p}\right) e\left(\frac{-\bar{4}(4v + b_1^2 + b_2^2 + b_3^2)\bar{z} - r_1 y \bar{z} + r_2 m \bar{y}}{p}\right).$$

If $p \nmid m$, $p|4v + b_1^2 + b_2^2 + b_3^2$, changing variable $y\bar{z} \rightarrow z$, we have

$$\begin{aligned} \mathcal{T}(b_1, b_2, b_3, r_1, r_2 m; p) &= \epsilon_p^3 p^{\frac{3}{2}} \sum_{y \in \mathbb{F}_p^\times} \left(\frac{y}{p}\right) e\left(\frac{r_2 m \bar{y}}{p}\right) \sum_{z \in \mathbb{F}_p^\times} \left(\frac{\bar{z}}{p}\right) e\left(\frac{-r_1 z}{p}\right) \\ &= \epsilon_p^3 p^{\frac{3}{2}} \left(\frac{r_2 m}{p}\right) \left(\frac{-\bar{r}_1}{p}\right) \tau\left(\left(\frac{\cdot}{p}\right)\right)^2, \end{aligned}$$

where $\tau\left(\left(\frac{\cdot}{p}\right)\right)$ is the Gauss sum associated with the quadratic residue $\left(\frac{\cdot}{p}\right)$, i.e.

$$\tau\left(\left(\frac{\cdot}{p}\right)\right) = \sum_{\gamma \bmod p} \left(\frac{\gamma}{p}\right) e\left(\frac{\gamma}{p}\right) = \epsilon_p \sqrt{p}.$$

Therefore,

$$\mathcal{T}(b_1, b_2, b_3, r_1, r_2 m; p) \ll p^{\frac{5}{2}}.$$

If $p \nmid m$, $p \nmid 4v + b_1^2 + b_2^2 + b_3^2$, we denote $r_0 = -\bar{4}(4v + b_1^2 + b_2^2 + b_3^2)$ and let $f(y, z) = r_0 z^{-1} - r_1 z^{-1} y + r_2 m y^{-1} \in \mathbb{F}_p^\times[y, z, (yz)^{-1}]$. The Newton polyhedron $\Delta(f)$ of f is the triangle in \mathbb{R}^2 with vertices $(-1, 0)$, $(-1, 1)$ and $(0, -1)$. Thus $\dim \Delta(f) = 2$. Moreover, for each of the following six polynomials

$$f_\sigma(y, z) = r_0 z^{-1}, -r_1 z^{-1} y, r_2 m y^{-1}, r_0 z^{-1} - r_1 z^{-1} y, r_0 z^{-1} + r_2 m y^{-1}, -r_1 z^{-1} y + r_2 m y^{-1}$$

corresponding to the faces of $\Delta(f)$ not containing $(0, 0)$, the locus of

$$\frac{\partial f_\sigma}{\partial y} = \frac{\partial f_\sigma}{\partial z} = 0$$

is empty in $\left(\overline{\mathbb{F}_p}^\times\right)^2$. In other words f is non-degenerate with respect to $\Delta(f)$. By Corollary 0.3 in Fu [2], we have

$$\sum_{y, z \in \mathbb{F}_p^\times} \left(\frac{z}{p}\right) e\left(\frac{r_0 \bar{z} - r_1 \bar{z} y + r_2 m \bar{y}}{p}\right) \ll p.$$

This completes the proof of Lemma 9.1. □

ACKNOWLEDGEMENTS. The authors would like to thank Professor Lei Fu for valuable discussion on the twisted character sums. The first author is supported by the National Natural Science Foundation of China (Grant No. 11101239) and the second author is supported by the Natural Science Foundation of Shandong Province (Grant No. ZR2015AM010).

REFERENCES

- [1] C. Calderón and M. J. de Velasco, *On divisors of a quadratic form*, Bol. Soc. Brasil. Mat. 31 (2000), 81-91.
- [2] L. Fu, *Weights of twisted exponential sums*, Math. Z. 262 (2009), 449-472.
- [3] J. B. Friedlander and H. Iwaniec, *A polynomial divisor problem*, J. reine angew. Math. 601 (2006), 109-137.
- [4] R. Guo and W. G. Zhai, *Some problems about the ternary quadratic form $m_1^2 + m_2^2 + m_3^2$* , Acta Arith. 156 (2012), 101-121.
- [5] D. R. Heath-Brown, *Cubic forms in ten variables*, Proc. London Math. Soc. 3 (2) (1983), 225-257.
- [6] C. Hooley, *On the number of divisors of quadratic polynomials*, Acta Math. 110 (1963): 97-114.
- [7] M. N. Huxley, *Area, lattice points, and exponential sums*, Oxford University Press, 1996.

- [8] Iv A. Ivić, *On the ternary additive divisor problem and the six moment of the zeta-function*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), 205-243, London Math. Soc. Lecture Note Ser., 237, Cambridge Univ. Press, Cambridge, 1997.
- [9] Iw H. Iwaniec, *Topics in classical automorphic forms*, American Mathematical Soc., 1997.
- [10] X. Li, *The central value of the Rankin-Selberg L -functions*, GAFA 18 (2009), 1660-1695.
- [11] X. Li, *The Voronoi formula for the triple divisor function*, Automorphic Forms and L -functions, ALM 30, 69-90.
- [12] Q. F. Sun, *Shifted convolution sums of GL_3 cusp forms with θ -series*, arXiv:1509.07644.
- [13] R. C. Vaughan, *The Hardy-Littlewood method*, Cambridge University Press, 1997.
- [14] X. M. Ren and Y. B. Ye, *Asymptotic Voronoi's summation formulas and their duality for $SL_3(\mathbb{Z})$* , Number theoryarithmetic in Shangri-La, 213C236, Ser. Number Theory Appl., 8, World Sci. Publ., Hackensack, NJ, 2013.
- [15] G. Yu, *On the number of divisors of the quadratic form $m^2 + n^2$* , Canad. Math. Bull. 43 (2) (2000), 239-256.
- [16] L. L. Zhao, *The sum of divisors of a quadratic form*, Acta Arith. 163 (2014), no. 2, 161C177.

QINGFENG SUN, SCHOOL OF MATHEMATICS AND STATISTICS, SHANDONG UNIVERSITY,
WEIHAI, WEIHAI, SHANDONG 264209, CHINA

E-mail address: qfsun@sdu.edu.cn

DEYU ZHANG, SCHOOL OF MATHEMATICAL SCIENCES, SHANDONG NORMAL UNIVERSITY,
JINAN, SHANDONG 250014, CHINA

E-mail address: zdy_78@hotmail.com